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ON SOME DIOPHANTINE EQUATIONS RELATED TO SQUARE TRIANGULAR AND BALANCING NUMBERS

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Abstract

The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the numbers. Undoubtely, triangular numbers are one of these numbers. In this study, we deal with the relations between square triangular numbers, special condition of triangular numbers, and balancing numbers. Also, we investigate positive integer solutions of some Diophantine equations such as $(x+y-1)^2 = 8xy$, $(x+y+1)^2 = 8xy$, $(x+y)^2 = 4x(2y \mp 1)$, $(x+y)^2 = 2x(4y \mp 1)$, $(x+y \mp 1)^2 = 8xy+1$, $x^2+y^2-6xy = \mp 1$, $x^2+y^2-6xy \mp x = 0$, $x^2-6xy+y^2 \mp 4x-1 = 0$, and other similar equations related to square triangular and balancing numbers.

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1. Introduction

By a triangular number, we mean the number of the form $T_n = n(n+1)/2,$ where n is a natural number. A few of these numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ..., and so on. Also, it is well known that x is a triangular number, if and only if $8x+1$ is a perfect square. The n-th triangular number is formed using an outher triangle, whose sides have n dots. Similarly, square numbers can be arranged in the shape of a square. The m-th square number is formed using an outher square, whose sides have m dots. The m-th square number is $S_m = m^2$[14]. Balancing numbers are numbers that are the solutions of the equation

$$1 + 2 + \dots + (m-1) = (m+1) + (m+2) + \dots + (m+r), \tag{1.1}$$

calling $r \in \mathbb{Z}^+,$ the balancer corresponding to the balancing number m. For example, 1, 6, 35, and 204 are balancing numbers with balancers 0, 2, 14, and 84, respectively. In what follows, we introduce cobalancing numbers in a way similar to the balancing numbers. By modifying (1.1), we call $m \in \mathbb{Z}^+,$ a cobalancing number if

$$1 + 2 + \dots + (m-1) + m = (m+1) + (m+2) + \dots + (m+r), \tag{1.2}$$

for some $r \in \mathbb{Z}^+.$ Here, we call $r \in \mathbb{Z}^+,$ a cobalancer corresponding to the cobalancing number m. A few of the cobalancing numbers are 0, 2, 14, and 84 with cobalancers 1, 6, 35, and 204, respectively [7], [8]. In this chapter, one of the major question we will be interested in answering is whether or not there is a close relations between square triangular numbers and balancing numbers. Since triangular numbers are of the form $T_n = \frac{n(n+1)}{2}$ and square numbers are of the form $S_m = m^2,$ square triangular numbers are integer solutions of the equation $m^2 = \frac{n(n+1)}{2}$ [14]. We will get this equation again using an amusing problem to see how square triangular numbers are obtained from this problem. In Equation (1.1), if we make substitution $m + r = n,$ we get

$$1 + 2 + \dots + (m - 1) = (m + 1) + (m + 2) + \dots + n. \tag{1.3}$$

Then this equation gives us a problem as follows:

I live on a street, whose houses are numbered in order $1, 2, 3, \dots, n - 1, n;$ so the houses at the ends of the street are numbered 1 and n. My own house number is m and of course, $0 < m < n.$ One day, I add up the house numbers of all the houses to the left of my house; then I do the same for all the houses to the right of my house. I find that the sums are the same. So, how can we find m and n [13]? Since $1 + 2 + 3 + \dots + m - 1 = (m + 1) + \dots + (n - 1) + n,$ it follows that

$$\frac{(m-1)m}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

Thus, we get $m^2 = \frac{n(n+1)}{2}$. Here, m^2 is both a triangular number and a square number. That is, m^2 is a square triangular number. In Equation (1.3), since m is a balancing number, it is easy to see that a balancing number is square root of a square triangular number. For more information about triangular, square triangular, and balancing numbers, see [1], [2], [12], and [17].

Euler [2] showed that $\left(\frac{a-b}{4\sqrt{2}}\right)^2$ is a triangular number for $a = (3 + 2\sqrt{2})^n$ and $b = (3 - 2\sqrt{2})^n$. In 1879 Roberts [2], by using Euler's formula, showed that the n-th square triangular number t_n is given by

$$t_n = \left[\frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{4\sqrt{2}}\right]^2.$$

After then Subramaniam showed that $u_n = 6u_{n-1} - u_{n-2}$ by taking $u_n = \sqrt{t_n}$ (see [15], [16]). Actually, the elements of the sequence (u_n) are balancing numbers, which are obtained from the solutions of the Equation (1.3). In this chapter, we will develop a method for finding all square triangular numbers after using well known theorems. For this, we will use some Diophantine equations, whose solutions are related to Pell, Pell-Lucas, and the sequence (v_n), where v_n is given by

$$v_n = Q_{2n}.$$

Before discussing about these sequences, we introduce two kinds of sequences named generalized Fibonacci and Lucas sequences $\{U_n\}$ and $\{V_n\}$, respectively. For more information about generalized Fibonacci and Lucas sequences, one can consult [4], [5], [6], [10], [11], and [18]. The generalized Fibonacci sequence $\{U_n\}$ with parameter k and t, is defined by $U_0 = 0, U_1 = 1,$ and $U_{n+1} = kU_n + tU_{n-1}$ for $n \geqslant 1$. Similarly, the generalized Lucas sequence $\{V_n\}$ with parameter k and t, is defined by $V_0 = 2, V_1 = k,$ and $V_{n+1} = kV_n + tV_{n-1}$ for $n \geqslant 1$, where $k^2 + 4t > 0$. Also generalized Fibonacci and Lucas numbers for negative subscript are defined as $U_{-n} = \frac{-U_n}{(-t)^n}$ and $V_{-n} = \frac{V_n}{(-t)^n}$ for all $n \in \mathbb{N}$. When $k = 2$ and $t = 1$, we get $U_n = P_n$ and $V_n = Q_n$, where P_n and Q_n are called *Pell and Pell-Lucas sequences*, respectively. Thus, $P_0 = 0, P_1 = 1,$ and $P_{n+1} = 2P_n + P_{n+1}$ for $n \geqslant 1$ and $Q_0 = 2, Q_1 = 2,$ and $Q_{n+1} = 2Q_n + Q_{n-1}$ for all $n \in \mathbb{N}$. When $k = 6$ and $t = -1$, we represent U_n and V_n by u_n and v_n, respectively. Thus, $u_0 = 0, u_1 = 1,$ and $u_{n+1} = 6u_n - u_{n-1}$ and $v_0 = 2, v_1 = 6,$ and $v_{n+1} = 6v_n - v_{n-1}$ for all $n \geqslant 1$. Now, we present some well known theorems regarding the sequences $(P_n), (Q_n),$ and (u_n) without proof.

Theorem 1.1. *Let* γ *and* δ *be roots of the characteristic equation* $x^2 - 2x - 1 = 0$. *Then, we have* $P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}}$ *and* $Q_n = \gamma^n + \delta^n$.

Theorem 1.2. *Let* α *and* β *be roots of the characteristic equation* $x^2 - 6x + 1 = 0$. *Then* $u_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}$ *and* $v_n = \alpha^n + \beta^n$.

The formulas given in the above theorems are known as Binet's formula. Let B_n denote n-th balancing number. From [8], we know that

$$B_n = \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}}.$$

From Theorems 1.1 and 1.2, it is easily seen that $u_n = B_n = \frac{P_{2n}}{2}$ and $v_n = Q_{2n}$.

The proof of the following theorem is given in [3].

Theorem 1.3. *Let* $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ *and* $\gamma = 1 + \sqrt{2}$. *Then the set of units of the ring* $\mathbb{Z}[\sqrt{2}]$ *is* $\{\mp \gamma^n | n \in \mathbb{Z}\}$.

Now, we mention some equations, whose solutions are related to (P_n) and (Q_n). Before discussing the equations, we give some known identities for (P_n), (Q_n), (B_n), and (v_n).

Well known identities for (P_n), (Q_n), (B_n), and (v_n) are

$$Q_n^2 - 8P_n^2 = 4(-1)^n, \tag{1.4}$$

$$v_n^2 - 32B_n^2 = 4, \tag{1.5}$$

$$P_{n+1} + P_{n-1} = Q_n, \tag{1.6}$$

$$B_n^2 - 6B_nB_{n-1} + B_{n-1}^2 = 1, \tag{1.7}$$

and

$$v_n^2 = v_{2n} + 2. \tag{1.8}$$

The following theorem is a well known theorem. We will give its proof for the sake of completeness.

Theorem 1.4. *All positive integer solutions of the Pell equation* $x^2 - 2y^2 = \mp 1$ *are given by* $(x, y) = \left(\frac{Q_n}{2}, P_n\right)$ *with* $n \geqslant 1$.

Proof. Assume that $(x, y) = \left(\frac{Q_n}{2}, P_n\right)$. Then $(x - \sqrt{2}y)(x + \sqrt{2}y) = \mp 1$ and this shows that $x - \sqrt{2}y$ is a unit in $\mathbb{Z}[\sqrt{2}]$. Moreover, since $x > 0$ and $y > 0$, we get $x + \sqrt{2}y > 1$. Therefore, there exists positive integer n such that $x + \sqrt{2}y = \gamma^n = \gamma P_n + P_{n-1}$ by Theorem 1.3. Since $\gamma P_n + P_{n-1}$ $= (1 + \sqrt{2})P_n + P_{n-1} = P_n + P_{n-1} + \sqrt{2}P_n$, we get $(x, y) = (P_n + P_{n-1}, P_n)$. Thus, $x = P_n + P_{n-1} = \frac{1}{2}(2P_n + 2P_{n-1}) = \frac{1}{2}(2P_n + P_{n-1} + P_{n-1}) = \frac{1}{2}(P_{n+1} + P_{n-1}) = \frac{1}{2}Q_n$ by identity (1.6). This shows that $x = \frac{Q_n}{2}$ and $y = P_n$. Conversely, if $(x, y) = (\frac{Q_n}{2}, P_n)$, then it follows from identity (1.4) that $x^2 - 2y^2 = \mp 1$. ∎

By using (1.4) and the above theorem, we can give the following corollaries:

Corollary 1. *All positive integer solutions of the Pell equation* $x^2 - 2y^2 = 1$ *are given by* $(x, y) = \left(\frac{Q_{2n}}{2}, P_{2n}\right)$ *with* $n \geqslant 1$.

Corollary 2. *All positive integer solutions of the Pell equation* $x^2 - 2y^2 = -1$ *are given by* $(x, y) = \left(\frac{Q_{2n+1}}{2}, P_{2n+1}\right)$ *with* $n \geqslant 0$.

We now give the characterization of all square triangular numbers.

Theorem 1.5. *A natural number x is a square triangular number, if and only if* $x = B_n^2$, *for some natural number n.*

Proof. Assume that x is square triangular number. Then $x = \frac{n(n+1)}{2} = m^2$, for some natural numbers n and m. Next, multiplying both sides by 8 and rewriting the previous equation, we obtain

$$8m^2 = 4n^2 + 4n = (2n+1)^2 - 1.$$

Let $x = 2n + 1$ and $y = 2m$. Then we get

$$x^2 - 2y^2 = 1.$$

Then by Corollary 1, it follows that $(x, y) = \left(\frac{Q_{2n}}{2}, P_{2n}\right) = \left(\frac{v_n}{2}, 2B_n\right)$.

Therefore, $2n + 1 = \frac{v_n}{2}$ and $m = B_n$. This shows that $x = m^2 = B_n^2$. Now assume that $x = B_n^2$. By identity (1.5), we get

$$x = B_n^2 = \frac{v_n^2 - 4}{32} = \frac{((v_n - 2)/4)((v_n + 2)/4)}{2} = \frac{((v_n - 2)/4)((v_n - 2)/4 + 1)}{2}.$$

Therefore, when $x = B_n^2$, x is a square triangular number. This concludes the proof. ■

2. Positive Solutions of Some Diophantine Equations

In this section, we consider the equations $(x + y - 1)^2 = 8xy$, $(x + y + 1)^2 = 8xy$, $(x + y)^2 = 4x(2y \mp 1)$, $(x + y \mp 1)^2 = 8xy + 1$, $x^2 + y^2 - 6xy = \mp 1$, $x^2 + y^2 - 6xy \mp x = 0$, and other similar equations. The solutions of these equations are related to square triangular numbers, balancing and cobalancing numbers, and the sequence (y_n), where y_n is given by

$$B_n^2 = \frac{y_n(y_n + 1)}{2}.$$

By Theorem 1.2, it is known that $B_n^2 = \frac{((v_n - 2)/4)((v_n + 2)/4)}{2}$. If we make substitution $\frac{v_n - 2}{4} = y_n$, then we get $B_n^2 = \frac{y_n(y_n + 1)}{2}$. Here, a few terms of (y_n) sequence are 1, 8, 49, 288, ..., and so on. The following lemma is given in [9]:

Lemma 1. *For* $n \geqslant 2$, $y_{n+1} = 6y_n - y_{n-1} + 2$.

Proof. Since $y_n = \dfrac{v_n - 2}{4}$ and $v_n = 6v_{n-1} - v_{n-2}$ with $n \geqslant 2$, we get

$$\begin{aligned} 6y_n - y_{n-1} + 2 &= 6((v_n - 2)/4) - ((v_{n-1} - 2)/4) + 2 \\ &= ((6v_n - v_{n-1} - 2)/4) \\ &= ((v_{n+1} - 2)/4) = y_{n+1}. \end{aligned}$$

That is, $y_{n+1} = 6y_n - y_{n-1} + 2.$ ∎

For $n = 1, 2, \ldots,$ let b_n be n-th cobalancing number and so let (b_n) denote the cobalancing number sequence. Then, the cobalancing numbers satisfy the similar recurrence relation given in Lemma 1. The proof of the following lemma is given in [7].

Lemma 2. *For* $n \geqslant 1$, $b_{n+1} = 6b_n - b_{n-1} + 2.$

Lemma 3. *For every* $n \geqslant 1$, $y_{2n} = 8B_n^2$ *and* $y_{2n+1} = 8B_nB_{n+1} + 1.$

Proof. By identities (1.5) and (1.8), we get

$$B_n^2 = \frac{v_n^2 - 4}{32} = \frac{v_{2n} - 2}{32} = \frac{y_{2n}}{8}.$$

Therefore $y_{2n} = 8B_n^2$. Moreover, since $y_{n+1} = 6y_n - y_{n-1} + 2,$ we get $y_n = (y_{n+1} + y_{n-1} - 2)/6$ and therefore by using $y_{2n} = 8B_n^2$, we find

$$y_{2n+1} = \frac{y_{2n+2} + y_{2n} - 2}{6} = \frac{8B_{n+1}^2 + 8B_n^2 - 2}{6} = \frac{8(B_{n+1}^2 + B_n^2) - 2}{6}.$$

Since $B_{n+1}^2 + B_n^2 = 6B_nB_{n+1} + 1$ by identity (1.7), we get

$$y_{2n+1} = \frac{8(B_{n+1}^2 + B_n^2) - 2}{6} = \frac{8(6B_nB_{n+1} + 1) - 2}{6} = 8B_nB_{n+1} + 1.$$

Therefore $y_{2n+1} = 8B_nB_{n+1} + 1.$ This concludes the proof. ∎

Now, we can give the following theorem. Since its proof is easy, we can omit it.

Theorem 2.1. *If n is an odd natural number, then* $y_n = \frac{Q_n^2}{4}$ *and if n is an even natural number, then* $y_n = \frac{Q_n^2}{4} - 1.$

At the introduction of this study, we say that if x is a triangular number, then $8x + 1$ is a perfect square. Also, since $y_{2n+1} = 8B_nB_{n+1} + 1$ and $y_{2n+1} = \frac{Q_{2n+1}^2}{4}$, it follows that B_nB_{n+1} is a triangular number. Moreover by Lemma 3, it is seen that y_n is odd iff n is odd and y_n is even iff n is even. From now on, we will try to find all positive integer solutions of some Diophantine equations. Before giving these equations, we present the following identities that will be useful for finding solutions. Since the proof of the identities is found by using Binet's formulas, Lemma 2, and induction method, we omit them.

Theorem 2.2. *For every natural number n, we have the following identities*:

$$v_{n+1} = 4P_{2n+1} + Q_{2n+1}, \tag{2.1}$$

$$v_n = 4P_{2n+1} - Q_{2n+1}, \tag{2.2}$$

$$B_{n+1} = \frac{Q_{2n+1} + 2P_{2n+1}}{4}, \tag{2.3}$$

$$B_n = \frac{Q_{2n+1} - 2P_{2n+1}}{4}, \tag{2.4}$$

$$b_n = y_n + B_n, \tag{2.5}$$

and

$$b_n = y_{n+1} - B_{n+1}. \tag{2.6}$$

In [9], Potter showed that $(y_n + y_{n+1} - 1)^2 = 8y_ny_{n+1}$. By means of this equation, we find other similar equations and we investigate solutions of them. Now we give the following theorem:

Theorem 2.3. *All positive integer solutions of the Diophantine equation* $(x + y - 1)^2 = 8xy$ *are given by* $(x, y) = (y_n, y_{n+1})$ *with* $n \geqslant 1$.

Proof. We know that $(y_n + y_{n+1} - 1)^2 = 8y_n y_{n+1}$ from [9]. Therefore, $(x, y) = (y_n, y_{n+1})$ is a solution of the equation $(x + y - 1)^2 = 8xy$. Now assume that $(x + y - 1)^2 = 8xy$ for some positive integers x and y. A simple computation shows that $x \neq y$. Then without loss of generality, we may suppose that $y > x$. If we make substitution $u = x + y$ and $v = y - x$, then we get $(u - 1)^2 = 2(u^2 - v^2)$ and therefore $u^2 - 2u + 1 = 2u^2 - 2v^2$. This shows that $v^2 - (\frac{u+1}{2})^2 = -1$. Then by Corollary 2, it follows that $(v, \frac{u+1}{2}) = (\frac{Q_{2n+1}}{2}, P_{2n+1})$ for some $n \geqslant 0$. Therefore, $u = 2P_{2n+1} - 1$ and $v = \frac{Q_{2n+1}}{2}$. Since $u = x + y$ and $v = y - x$, we get $x = \frac{u-v}{2}$ and $y = \frac{u+v}{2}$. This shows that $x = \frac{4P_{2n+1} - Q_{2n+1} - 2}{4}$ and $y = \frac{4P_{2n+1} + Q_{2n+1} - 2}{4}$. By using identities (2.1) and (2.2), we get $x = \frac{v_n - 2}{4}$ and $y = \frac{v_{n+1} - 2}{4}$. This implies that $x = y_n$ and $y = y_{n+1}$. ■

By using the above theorem, we can give the following three theorems in a similar way:

Theorem 2.4. *All positive integer solutions of the Diophantine equation* $(x + y - 2)^2 = 8xy$ *are given by* $(x, y) = (2y_n, 2y_{n+1})$ *with* $n \geqslant 1$.

Theorem 2.5. *All positive integer solutions of the Diophantine equation* $(x + y + 1)^2 = 8xy$ *are given by* $(x, y) = (y_n + 1, y_{n+1} + 1)$ *with* $n \geqslant 0$.

Proof. Assume that $(x+y+1)^2 = 8xy$ for some positive integers x and y. Let we make substitution $u = x-1$ and $v = y-1$. Then we get

$$(u+v+3)^2 = 8(u+1)(v+1).$$

When we rearrange the equation, we get $(u+v-1)^2 = 8uv$. Then the proof follows from Theorem 2.3. Conversely, if $(x, y) = (y_n + 1, y_{n+1} + 1)$, then a simple computation shows that $(x+y+1)^2 = 8xy$. ∎

Theorem 2.6. *All positive integer solutions of the Diophantine equation* $(x+y+2)^2 = 8xy$ *are given by* $(x, y) = (2y_n + 2, 2y_{n+1} + 2)$ *with* $n \geqslant 0$.

Now, we can give the following theorems, formed by means of the above theorems, whose solutions are related to square triangular numbers, balancing and cobalancing numbers, and the sequence (y_n).

Theorem 2.7. *All positive integer solutions of the equation* $(x+y)^2 = 4x(2y+1)$ *are given by* $(x, y) = (4B_n^2, 4B_nB_{n+1})$ *or* $(x, y) = (4B_{n+1}^2, 4B_nB_{n+1})$ *with* $n \geqslant 1$.

Proof. Assume that $(x+y)^2 = 4x(2y+1)$ for some positive integers x and y. Then $(2x+2y)^2 = 16x(2y+1)$ and therefore $(2x+2y+1-1)^2 = 8.2x(2y+1)$. Then by Theorem 2.3, it follows that $(2x, 2y+1) = (y_n, y_{n+1})$ or $(2x, 2y+1) = (y_{n+1}, y_n)$ for some $n \geqslant 1$. Firstly, assume that $(2x, 2y+1) = (y_n, y_{n+1})$ for some $n \geqslant 1$. Since y_n is even, then n is also even. Let $n = 2k$ with $k \geqslant 1$. Thus $x = y_{2k}/2$ and $y = y_{2k+1} - 1/2$. By using Lemma 3, it follows that $x = 4B_k^2$ and $y = 4B_kB_{k+1}$. Similarly, if $(2x, 2y+1) = (y_{n+1}, y_n)$, we see that $x = 4B_{k+1}^2$ and $y = 4B_kB_{k+1}$. Conversely, if $(x, y) = (4B_n^2, 4B_nB_{n+1})$ or $(x, y) = (4B_{n+1}^2, 4B_nB_{n+1})$, then with a simple computation by using identity (1.6), we get $(x+y)^2 = 4x(2y+1)$. ∎

From the above theorem, we can give the following corollary:

Corollary 3. *All positive integer solutions of the equation* $x^2 + y^2 - 6xy - 2x = 0$ *are given by* $(x, y) = (2B_n^2, 2B_nB_{n+1})$ *or* $(x, y) = (2B_{n+1}^2, 2B_nB_{n+1})$ *with* $n \geqslant 1$.

Proof. $x^2 + y^2 - 6xy - 2x = 0$ iff $(x + y)^2 = 2x(4y + 1)$ iff $(2x + 2y)^2 = 4.2x(2.2y + 1)$. Then the proof follows from Theorem 2.7. ■

Since the proof of the following theorem is similar to that of above theorem, we omit it.

Theorem 2.8. *All positive integer solutions of the equation* $(x + y)^2 = 4x(2y - 1)$ *are given by* $(x, y) = (4B_kB_{k+1} + 1, 4B_k^2 + 1)$ *or* $(x, y) = (4B_kB_{k+1} + 1, 4B_{k+1}^2 + 1)$ *with* $k \geqslant 1$.

From the above theorem, we can give the following corollary without proof:

Corollary 4. *There is no positive integer solutions of the equation* $x^2 + y^2 - 6xy + 2x = 0$.

Theorem 2.9. *All positive integer solutions of the equation* $(x + y + 1)^2 = 8xy + 1$ *are given by* $(x, y) = (1, 1)$ *or* $(x, y) = (b_n + 1, b_{n-1} + 1)$ *with* $n \geqslant 1$.

Proof. Assume that $(x + y + 1)^2 = 8xy + 1$ for some positive integers x and y. For $x = y$, it is clear that $(x, y) = (1, 1)$ is a solution of the equation $(x + y + 1)^2 = 8xy + 1$. Then assume that $x \neq y$. Then without loss of generality, we may suppose that $x > y$. Let $u = x + y$ and $v = x - y$. Then $x = \frac{u + v}{2}$ and $y = \frac{u - v}{2}$. Since $(x + y + 1)^2 = 8xy + 1$, we get $(u + 1)^2 = 8(\frac{u^2 - v^2}{4}) + 1$. This shows that $(u - 1)^2 - 2v^2 = 1$. By

Corollary 1, it is seen that $(u-1, v) = \left(\frac{v_n}{2},\ 2B_n\right)$ for some $n \geqslant 1$. Thus $u = \frac{v_n}{2} + 1$ and $v = 2B_n$. Substituting these values of u and v into the equalities $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, we get

$$x = \frac{u+v}{2} = \frac{v_n/2 + 1 + 2B_n}{2} = \frac{v_n + 4B_n + 2}{4} = \frac{v_n - 2 + 4B_n + 4}{4}$$

$$= y_n + B_n + 1,$$

and

$$y = \frac{u-v}{2} = \frac{v_n/2 + 1 - 2B_n}{2} = \frac{v_n - 4B_n + 2}{4} = \frac{v_n - 2 - 4B_n + 4}{4}$$

$$= y_n - B_n + 1.$$

That is, $x = y_n + B_n + 1$ and $y = y_n - B_n + 1$. By identities (2.5) and (2.6), it follows that $x = b_n + 1$ and $y = b_{n-1} + 1$. Conversely, if $x = b_n + 1$ and $y = b_{n-1} + 1$, then a simple computation shows that $(b_n + b_{n-1} + 3)^2 = 8(b_{n-1} + 1)(b_n + 1) + 1$. This concludes the proof. ■

Since the proof of the following theorem is similar to that of above theorem, we omit it.

Theorem 2.10. *All positive integer solutions of the equation* $(x + y - 1)^2 = 8xy + 1$ *are given by* $(x, y) = (b_n, b_{n-1})$ *with* $n \geqslant 2$.

Theorem 2.11. *All positive integer solutions of the equation* $x^2 + y^2 - 6xy = 1$ *are given by* $(x, y) = (B_{n+1}, B_n)$ *with* $n \geqslant 1$.

Proof. Assume that $x^2 + y^2 - 6xy = 1$ for some positive integers x and y. Then $(x - y)^2 - 4xy = 1$ and therefore $x \neq y$. Without loss of generality, we may suppose that $x > y$. If we make substitution $u = x + y$ and $v = x - y$, we get $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Since $x^2 + y^2 - 6xy - 1 = 0$, it follows that $(x - y)^2 - 4xy = 1$. Thus, if we rewrite new values of x and y into the equation $(x-y)^2 - 4xy = 1$, we get $v^2 - (u^2 - v^2) = 1$. This shows that $u^2 - 2v^2 = -1$. By Corollary 2, it follows that $(u, v) = (\frac{Q_{2n+1}}{2}, P_{2n+1})$ for some $n \geqslant 0$. Therefore $u = \frac{Q_{2n+1}}{2}$ and $v = P_{2n+1}$. Since $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, it is seen that $x = \frac{Q_{2n+1} + 2P_{2n+1}}{4}$ and $y = \frac{Q_{2n+1} - 2P_{2n+1}}{4}$. By using identities (2.3) and (2.4), we get $x = B_{n+1}$ and $y = B_n$. Conversely, if $(x, y) = (B_{n+1}, B_n)$, then by identity (1.7), it follows that $x^2 + y^2 - 6xy = 1$. This concludes the proof. ■

Theorem 2.12. *There is no positive integer solutions of the equation* $x^2 + y^2 - 6xy = -1$.

Proof. Assume that $x^2 + y^2 - 6xy = -1$. Then $(x-y)^2 - 4xy = -1$ and therefore $x - y$ is an odd integer. Let $x + y = u$ and $x - y = v$. Then, it can be seen that $u^2 - 2v^2 = 1$. Since v is an odd integer, then $u^2 = 2v^2 + 1 \equiv 3 \pmod 8$, a contradiction. This concludes the proof. ■

Theorem 2.13. *All positive integer solutions of the equation* $x^2 + y^2 - 6xy - x = 0$ *are given by* $(x, y) = (B_k^2, B_k B_{k+1})$ *or* $(x, y) = (B_{k+1}^2, B_k B_{k+1})$ *with* $k \geqslant 1$.

Proof. Assume that $x^2 + y^2 - 6xy - x = 0$ for some positive integers x and y. Then $(4x)^2 + (4y)^2 - 6(4x)(4y) - 4(4x) = 0$. Let $4x = a$ and $4y = b$. Then, it can be seen that $a^2 + b^2 - 6ab - 4a = 0$. That is, $(a+b)^2 = 4a(2b+1)$. Thus, by Theorem 2.7, there exists $k \geqslant 1$ such that $(a, b) = (4B_k^2, 4B_k B_{k+1})$ or $(a, b) = (4B_{k+1}^2, 4B_k B_{k+1})$. Therefore, $a = 4B_k^2$, $b = 4B_k B_{k+1}$ or $a = 4B_{k+1}^2$, $b = 4B_k B_{k+1}$. Since $a = 4x$ and

$b = 4y$, it is seen that $x = B_k^2$, $y = B_k B_{k+1}$ or $x = B_{k+1}^2$, $y = B_k B_{k+1}$. Conversely, if $(x, y) = (B_k^2, 4B_k B_{k+1})$ or $(x, y) = (4B_{k+1}^2, 4B_k B_{k+1})$, it is easy to see that $x^2 + y^2 - 6xy - x = 0$. ∎

Theorem 2.14. *There is no positive integer solutions of the equation* $x^2 + y^2 - 6xy + x = 0$.

Proof. Assume that $x^2 + y^2 - 6xy + x = 0$ for some positive integers x and y. Then $(4x)^2 + (4y)^2 - 6(4x)(4y) + 4(4x) = 0$. Thus, by Theorem 2.8, there exists $k \geqslant 1$ such that $(4x, 4y) = (4B_k B_{k+1} + 1, 4B_k^2 + 1)$ or $(4x, 4y) = (4B_k B_{k+1} + 1, 4B_{k+1}^2 + 1)$, which is impossible. This concludes the proof. ∎

Actually, we produce many equations considering from Potter's [9] showing $(y_n + y_{n+1} - 1)^2 = 8y_n y_{n+1}$. Now we give other similar equations, whose solutions are related balancing and cobalancing numbers more. While solving these equations, we again use Pell equations.

Theorem 2.15. *All positive integer solutions of the equation* $(x + y - 1)^2 = 8xy + 4$ *are given by* $(x, y) = (B_n + b_n, B_n - b_{n-1} - 1)$ *with* $n \geqslant 1$.

Proof. Assume that $(x + y - 1)^2 = 8xy + 4$ for some positive integers x and y. Then it follows that $x \neq y$. Without loss of generality, we may suppose $x > y$. Let $u = x + y$ and $v = x - y$. Thus, we get $(u - 1)^2 = 2(u^2 - v^2) + 4$. When we rearrange the equation, it is seen that $v^2 - 2(\frac{u+1}{2})^2 = 1$. Thus by Corollary 1, there exists $n \geqslant 1$. such that $(v, \frac{u+1}{2}) = (\frac{Q_{2n}}{2}, P_{2n})$. Therefore $v = \frac{Q_{2n}}{2}$ and $u = 2P_{2n} - 1$. Since $\frac{Q_{2n}}{2} = \frac{v_n}{2}$ and $P_{2n} = 2B_n$, we can write $v = \frac{v_n}{2}$ and $u = 4B_n - 1$. On the other hand, since $u = x + y$ and $v = x - y$, it is seen that $x = \frac{u + v}{2}$ and $y = \frac{u - v}{2}$. Now, if we write the values of u and v into the equalities $x = \frac{u + v}{2}$ and $y = \frac{u - v}{2}$, we get $x = \frac{v_n - 2 + 8B_n}{4}$ and $y = \frac{8B_n - 4 - (v_n - 2)}{4}$. By identities (2.5), (2.6), and the equality $y_n = \frac{v_n - 2}{4}$, it follows that $x = B_n + b_n$ and $y = B_n - b_{n-1} - 1$. Conversely, if $(x, y) = (B_n + b_n, B_n - b_{n-1} - 1)$ with $n \geqslant 1$, then from identities (2.5), (2.6), and Lemma 3, it follows that $(x + y - 1)^2 = 8xy + 4$.

■

In a similar way, we can give the following theorem without proof:

Theorem 2.16. *All positive integer solutions of the equation* $(x + y + 1)^2 = 8xy + 4$ *are given by* $(x, y) = (B_n + b_n + 1, B_n - b_{n-1})$ *with* $n \geqslant 1$.

Finally, we give two theorems, whose solutions are interesting.

Theorem 2.17. *All positive integer solutions of the equation* $x^2 - 6xy + y^2 + 4x - 1 = 0$ *are given by*

$$(x, y) = \begin{cases} \left(\frac{B_n + b_n + 1}{2}, \frac{B_n - b_{n-1} + 1}{2}\right); & n \text{ is odd} \\ \left(\frac{B_n - b_{n-1}}{2}, \frac{B_n + b_n + 2}{2}\right); & n \text{ is even} \end{cases}$$

with $n \geqslant 1$.

Proof. Assume that $x^2 - 6xy + y^2 + 4x - 1 = 0$. Then, $x^2 - 6xy + y^2 + 4x - 1 = 0$ iff $(x + y)^2 = 4x(2y - 1) + 1$ for some natural numbers x and y. Next, when we multiply both sides of the equation $(x + y)^2 = 4x(2y - 1) + 1$ by 4 and rewrite previous equation, we get

$$(2x + 2y - 1 + 1)^2 = 8.2x(2y - 1) + 4.$$

Let $u = 2x$ and $v = 2y - 1$. Then we get $(u + v + 1)^2 = 8uv + 4$. Since all positive integer solutions of the equation $(u + v + 1)^2 = 8uv + 4$ are $(u, v) = (B_n + b_n + 1, B_n - b_{n-1})$ or $(u, v) = (B_n - b_{n-1}, B_n + b_n + 1)$ by Theorem 2.16, it is seen that $u = B_n + b_n + 1$ and $v = B_n - b_{n-1}$ or $u = B_n - b_{n-1}$ and $v = B_n + b_n + 1$. Thus, we get $x = (B_n + b_n + 1)/2$ and $y = (B_n - b_{n-1} + 1)/2$ or $x = (B_n - b_{n-1})/2$ and $y = (B_n + b_n + 2)/2$. It is well known that B_n is even iff n is even and b_n is always even. Then we get

$$(x, y) = \begin{cases} \left(\dfrac{B_n + b_n + 1}{2}, \dfrac{B_n - b_{n-1} + 1}{2}\right); & n \text{ is odd} \\ \left(\dfrac{B_n - b_{n-1}}{2}, \dfrac{B_n + b_n + 2}{2}\right); & n \text{ is even} \end{cases}.$$

Conversely, if $(x, y) = \left(\dfrac{B_n + b_n + 1}{2}, \dfrac{B_n - b_{n-1} + 1}{2}\right)$, where n is odd and $(x, y) = \left(\dfrac{B_n - b_{n-1}}{2}, \dfrac{B_n + b_n + 2}{2}\right)$, where n is even; by using identities (2.5), (2.6), and Lemma 3, it can be shown that $x^2 - 6xy + y^2 + 4x - 1 = 0$. This concludes the proof. ∎

Now we can give the following theorem easily:

Theorem 2.18. *All positive integer solutions of the equation* $x^2 - 6xy + y^2 - 4x - 1 = 0$ *are given by*

$$(x, y) = \begin{cases} \left(\dfrac{B_n - b_{n-1} - 1}{2}, \dfrac{B_n + b_n - 1}{2}\right); & n \textit{ is odd} \\ \left(\dfrac{B_n + b_n}{2}, \dfrac{B_n - b_{n-1} - 2}{2}\right); & n \textit{ is even} \end{cases}$$

with $n \geqslant 2$.

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